

Nondegeneracy of the Lump Solution to the KP-I Equation

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1 Introduction

The KP equation, introduced by Kadomtsev and Petaviashvili [18], is a classical nonlinear dispersive equation appearing in many physical contexts, including the motion of shallow water waves. It has the form

$$\partial_x (\partial_t u + \partial_x^3 u + 3\partial_x (u^2)) - \sigma \partial_y^2 u = 0, \quad (1)$$

where $\sigma = \pm 1$. In this paper, we will consider the case of $\sigma = 1$, which is called KP-I equation. (When $\sigma = -1$, it is called KP-II equation). If u is y independent, then equation (1) reduces to the KdV equation, a well known integrable system. Dryuma [10] proved that the KP equation has a Lax pair. The inverse scattering transform for KP-I equation has been carried out in [11, 21, 31]. (See [1, 2] and the reference therein for more discussions on the development of this topic before 1991.)

The soliton solutions of the KP equation can be obtained by various different methods, both for the KP-I and KP-II equations. An important feature of the KP-I equation is that it has a family of lump solutions which are also travelling wave type. They were found in [22, 28]. Explicitly, the KP-I equation has a lump solution of the form $Q(x - t, y)$, where

$$Q(x, y) = 4 \frac{y^2 - x^2 + 3}{(x^2 + y^2 + 3)^2}. \quad (2)$$

Note that the function Q decays like $O(r^{-2})$ at infinity. (Here $r^2 = x^2 + y^2$.) Because of this slow decaying property, the inverse scattering transform of KP-I is quite delicate. Indeed, it is shown in [29] that a winding number could be associated to lump type solutions. The interaction of multi-lump solutions has been studied in [12, 20]. The lump solutions also appear in other physical models. For instance, formal asymptotic analysis shows that in the context of the motion in a Bose condensate, the transonic limit of the traveling wave solutions to the GP equation approach to the lump like solutions ([3, 17]). This is called Roberts's Program ([16, 17]). A rigorous verification is given recently in [4] and [6]. (For traveling waves of Gross-Pitaevskii equation, we refer to [5] and [23] and the references therein.)

Regarding to traveling wave type solutions, it is worth mentioning that the generalized KP-I equations

$$\partial_x^2 (\partial_x^2 u - u + u^p) - \partial_y^2 u = 0 \quad (3)$$

also have lump type solutions for suitable p ([7, 19]). For general p , this is not an integrable system. Hence no explicit formula is available. These lump type solutions are obtained via variational methods (concentration compactness). We point out that the uniqueness of these lump type solutions is an open problem and therefore it is not known whether Q has this variational characterization. The stability(instability) properties of these solutions have been studied in ([9, 30]). Formal computation in [30] suggests that the lump solutions Q should be stable.

Towards understanding the spectral and stability property of the lump solution, we shall prove in this paper that Q is nondegenerate (and hence locally unique up to translations). Our main result in this paper is

Theorem 1 *Suppose ϕ is a smooth solution to the equation*

$$\partial_x^2 (\partial_x^2 \phi - \phi + 6Q\phi) - \partial_y^2 \phi = 0. \quad (4)$$

Assume

$$\phi(x, y) \rightarrow 0, \text{ as } x^2 + y^2 \rightarrow +\infty.$$

Then $\phi = c_1 \partial_x Q + c_2 \partial_y Q$, for certain constants c_1, c_2 .

Theorem 1 has long been conjectured to be true. See the remark after Lemma 7 in [30] concerning the spectral property and its relation to stability. We also expect that the linearized operator around Q has exactly one negative eigenvalue. But we have not been able to prove this. The rigidity result, stated in Theorem 1, may be very useful in the construction and analysis of solutions near the KP-I equation.

Let us briefly describe the main ideas of the proof. Our main tool will be the Bäcklund transformation. It is known that the lump solution can be obtained from the trivial solution by performing Bäcklund transformation twice. We consider these transformation in the linearized level. We show that a kernel of the linearized operator around Q could be transformed to a kernel of the

operator $\partial_x^2 + \partial_y^2 - \partial_x^4$, which is the linearized operator around the trivial solution *zero*. With some information on the growth rate of the kernel function, we are able to conclude that the only decaying solutions to (4) are corresponding to translations in the x and y axes. The idea of using linearized Bäcklund transformation to investigate the spectral property has already been used in [25] in the case of Toda lattice, see also [24] for discussion in the case of N-solitons for the KdV equation.

The organization of the paper is as follows: In Section 2, we recall some basic facts about the bilinear derivative. In Section 3 and Section 4, we analyze the linearized Bäcklund transformation. In Section 5, we prove Theorem 1.

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2 Preliminaries

We will use D to denote the bilinear derivative. Explicitly,

$$D_x^m D_t^n f \cdot g = \partial_y^m \partial_s^n f(x+y, t+s) g(x-y, t-s) |_{s=0, y=0}. \quad (5)$$

In particular, it holds

$$\begin{aligned} D_x f \cdot g &= \partial_x f g - f \partial_x g, \\ D_x^2 f \cdot g &= \partial_x^2 f g - 2 \partial_x f \partial_x g + f \partial_x^2 g, \\ D_x D_t f \cdot g &= \partial_x \partial_t f g - \partial_x f \partial_t g - \partial_t f \partial_x g + f \partial_x \partial_t g. \end{aligned}$$

The KP-I equation (1) can be written in the following bilinear form [13]:

$$(D_x D_t + D_x^4 - D_y^2) \tau \cdot \tau = 0. \quad (6)$$

To explain this, we introduce the τ -function:

$$u = 2 \partial_x^2 (\ln \tau). \quad (7)$$

Then equation (1) becomes

$$\partial_x^2 \left[\left(\partial_x \partial_t \ln \tau + \partial_x^4 \ln \tau + 6 (\partial_x^2 \ln \tau)^2 \right) - \partial_y^2 \ln \tau \right] = 0.$$

Using the identities (see Section 1.7.2 of [13])

$$2 \partial_x^2 \ln \tau = \frac{D_x^2 \tau \cdot \tau}{\tau^2}, \quad 2 \partial_x \partial_t \ln \tau = \frac{D_x D_t \tau \cdot \tau}{\tau^2},$$

and

$$2 \partial_x^4 \ln \tau = \frac{D_x^4 \tau \cdot \tau}{\tau^2} - 3 \left(\frac{D_x^2 \tau \cdot \tau}{\tau^2} \right)^2,$$

we get

$$\partial_x^2 \left(\frac{D_x D_t \tau \cdot \tau + D_x^4 \tau \cdot \tau - D_y^2 \tau \cdot \tau}{\tau^2} \right) = 0.$$

Therefore, if τ satisfies the bilinear equation (6), then u satisfies the KP-I equation (1).

Let i be the imaginary unit. We will investigate properties of the following three functions:

$$\begin{aligned}\tau_0(x, y) &= 1, \\ \tau_1(x, y) &= x + iy + \sqrt{3}, \\ \tau_2(x, y) &= x^2 + y^2 + 3.\end{aligned}$$

Let

$$\begin{aligned}\tilde{\tau}_0(x, y, t) &= \tau_0(x - t, y), \\ \tilde{\tau}_1(x, y, t) &= \tau_1(x - t, y), \\ \tilde{\tau}_2(x, y, t) &= \tau_2(x - t, y).\end{aligned}$$

Then $\tilde{\tau}_0, \tilde{\tau}_1, \tilde{\tau}_2$ are solutions of (6).

2.1 The Bäcklund transformation

Under (7), the function $\tilde{\tau}_2$ is corresponding to the lump solution $Q(x - t, y)$. Note that the solution corresponding to $\tilde{\tau}_1$ is not real valued.

Now let μ be a constant. We first recall the following bilinear operator identity (see [26] or P. 90 in [27] for more details, we also refer to [15] and the references therein for the construction of more general rational solutions):

$$\begin{aligned}& \frac{1}{2} [(D_x D_t + D_x^4 - D_y^2) f \cdot f] g g - \frac{1}{2} [(D_x D_t + D_x^4 - D_y^2) g \cdot g] f f \\&= D_x \left[(D_t - \sqrt{3}i\mu D_y + D_x^3 - \sqrt{3}i D_x D_y) f \cdot g \right] \cdot (fg) \\&+ 3D_x \left[\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}}i D_y \right) f \cdot g \right] \cdot (D_x g \cdot f) \\&+ \sqrt{3}i D_y \left[\left(D_x^2 + \mu D_x + \frac{1}{\sqrt{3}}i D_y \right) f \cdot g \right] \cdot (fg).\end{aligned}\tag{8}$$

By this identity, we can consider the Bäcklund transformation from $\tilde{\tau}_0$ to $\tilde{\tau}_1$ ($\mu = \frac{1}{\sqrt{3}}$):

$$\begin{cases} \left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}i D_y \right) \tilde{\tau}_0 \cdot \tilde{\tau}_1 = 0, \\ (D_t - i D_y + D_x^3 - \sqrt{3}i D_x D_y) \tilde{\tau}_0 \cdot \tilde{\tau}_1 = 0. \end{cases}\tag{9}$$

The Bäcklund transformation from $\tilde{\tau}_1$ to $\tilde{\tau}_2$ is given by ($\mu = -\frac{1}{\sqrt{3}}$)

$$\begin{cases} \left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}i D_y \right) \tilde{\tau}_1 \cdot \tilde{\tau}_2 = 0, \\ (D_t + i D_y + D_x^3 - \sqrt{3}i D_x D_y) \tilde{\tau}_1 \cdot \tilde{\tau}_2 = 0. \end{cases}\tag{10}$$

Throughout the paper, we set $r = \sqrt{x^2 + y^2}$. We would like to relate the kernel of the linearized KP-I equation to that of the linearized equation in bilinear form.

Lemma 2 *Suppose ϕ is a smooth function satisfying the linearized KP-I equation*

$$\partial_x^2 (\partial_x^2 \phi - \phi + 6Q\phi) - \partial_y^2 \phi = 0.$$

Assume

$$\phi(x, y) \rightarrow 0, \text{ as } r \rightarrow +\infty.$$

Let

$$\eta(x, y) = \tau_2 \int_0^x \int_{-\infty}^t \phi(s, y) ds dt.$$

Then η satisfies

$$(-D_x^2 + D_x^4 - D_y^2) \eta \cdot \tau_2 = 0.$$

Moreover,

$$|\eta| + (|\partial_x \eta| + |\partial_y \eta| + |\partial_x^2 \eta| + |\partial_x \partial_y \eta| + |\partial_x^3 \eta|) (1 + r) \leq C (1 + r)^{\frac{5}{2}}. \quad (11)$$

Proof. We write the equation

$$\partial_x^2 (\partial_x^2 \phi - \phi + 6Q\phi) - \partial_y^2 \phi = 0$$

as

$$\partial_x^4 \phi - \partial_x^2 \phi - \partial_y^2 \phi = -6\partial_x^2 (Q\phi).$$

Since $\phi \rightarrow 0$ as $r \rightarrow +\infty$, it follows from the a priori estimate of the operator $\partial_x^2 - \partial_x^4 + \partial_y^2$ (see Lemma 3.6 of [8]) that

$$|\phi| + (|\partial_x \phi| + |\partial_y \phi|) (1 + r) \leq C (1 + r)^{-2}.$$

Moreover, $\int_{-\infty}^{+\infty} \phi(x, y) dx = 0$, for all y . Therefore,

$$|\eta| + (|\partial_x \eta| + |\partial_y \eta| + |\partial_x^2 \eta| + |\partial_x \partial_y \eta| + |\partial_x^3 \eta|) (1 + r) \leq C (1 + r)^{\frac{5}{2}}.$$

As a consequence,

$$\frac{-D_x^2 \eta \cdot \tau_2 + D_x^4 \eta \cdot \tau_2 - D_y^2 \eta \cdot \tau_2}{\tau_2^2} \rightarrow 0, \text{ as } r \rightarrow +\infty. \quad (12)$$

On the other hand, since ϕ satisfies the linearized KP-I equation, η satisfies

$$\partial_x^2 \left(\frac{-D_x^2 \eta \cdot \tau_2 + D_x^4 \eta \cdot \tau_2 - D_y^2 \eta \cdot \tau_2}{\tau_2^2} \right) = 0.$$

This together with (12) implies

$$-D_x^2 \eta \cdot \tau_2 + D_x^4 \eta \cdot \tau_2 - D_y^2 \eta \cdot \tau_2 = 0.$$

The proof is completed. ■

3 Linearized Bäcklund transformation between τ_0 and τ_1

In terms of τ_0 and τ_1 , the Bäcklund transformation (9) can be written as

$$\begin{cases} \left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y \right) \tau_0 \cdot \tau_1 = 0, \\ \left(-D_x - iD_y + D_x^3 - \sqrt{3}iD_xD_y \right) \tau_0 \cdot \tau_1 = 0. \end{cases} \quad (13)$$

Linearizing this system at (τ_0, τ_1) , we obtain

$$\begin{cases} L_1\phi = G_1\eta, \\ M_1\phi = N_1\eta. \end{cases} \quad (14)$$

Here for notational simplicity, we have defined

$$\begin{aligned} L_1\phi &= \left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y \right) \phi \cdot \tau_1, \\ M_1\phi &= \left(-D_x - iD_y + D_x^3 - \sqrt{3}iD_xD_y \right) \phi \cdot \tau_1, \end{aligned}$$

and

$$\begin{aligned} G_1\eta &= - \left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y \right) \tau_0 \cdot \eta, \\ N_1\eta &= - \left(-D_x - iD_y + D_x^3 - \sqrt{3}iD_xD_y \right) \tau_0 \cdot \eta. \end{aligned}$$

Proposition 3 *Let η be a solution of the linearized bilinear KP-I equation at τ_1 :*

$$-D_x^2\eta \cdot \tau_1 + D_x^4\eta \cdot \tau_1 - D_y^2\eta \cdot \tau_1 = 0. \quad (15)$$

Suppose η satisfies (11). Then the system (14) has a solution ϕ with

$$|\phi| + |\partial_x\phi| + |\partial_y\phi| \leq C(1+r)^{\frac{5}{2}}.$$

The rest of this section will be devoted to the proof of Proposition 3.

From the first equation in (14), we get

$$\partial_y\phi\tau_1 = i \left[\partial_x\phi\tau_1 + \sqrt{3}(\partial_x^2\phi\tau_1 - 2\partial_x\phi) \right] - \sqrt{3}iG_1\eta. \quad (16)$$

Inserting (16) into the right hand side of the second equation of (14), we get

$$4\partial_x^3\phi\tau_1 + \left(2\sqrt{3}\tau_1 - 12 \right) \partial_x^2\phi + \left(-4\sqrt{3} + \frac{12}{\tau_1} \right) \partial_x\phi = F_1. \quad (17)$$

Here

$$\begin{aligned} F_1 &= 3\partial_x(G_1\eta) + \sqrt{3}G_1\eta + N_1\eta - \frac{6}{\tau_1}G_1\eta \\ &= -2\partial_x^3\eta + 2\sqrt{3}i\partial_x\partial_y\eta - \frac{6}{\tau_1}G_1\eta. \end{aligned}$$

To solve the equation (17), we shall first analyze the solutions of the homogeneous equation

$$2\tau_1^2 g'' + (\sqrt{3}\tau_1 - 6) \tau_1 g' + (6 - 2\sqrt{3}\tau_1) g = 0. \quad (18)$$

Lemma 4 *The equation (18) has two linearly independent solutions*

$$\begin{aligned} g_1 &= \tau_1 - \frac{\sqrt{3}}{2} \tau_1^2, \\ g_2 &= \tau_1 e^{-\frac{\sqrt{3}}{2} \tau_1} \end{aligned}$$

Proof. This can be verified directly. Indeed, the equation (18) can be exactly solved using software such as *Mathematica*. We will frequently use this software in the rest of paper. ■

Let W be the Wronskian of g_1, g_2 . That is,

$$W = g_1 \partial_x g_2 - g_2 \partial_x g_1 = \frac{3}{4} \tau_1^3 e^{-\frac{\sqrt{3}}{2} \tau_1}.$$

By the variation of parameter formula, the equation

$$4\tau_1 g'' + (2\sqrt{3}\tau_1 - 12) g' + \left(\frac{12}{\tau_1} - 4\sqrt{3}\right) g = F_1$$

has a solution of the form

$$g^*(x, y) = g_2(x, y) \int_{-\infty}^x \frac{g_1 F_1}{4\tau_1 W} ds - g_1(x, y) \int_{-\infty}^x \frac{g_2 F_1}{4\tau_1 W} ds.$$

It follows that for each fixed y , the equation

$$4\tau_1 \phi''' + (2\sqrt{3}\tau_1 - 12) \phi'' + \left(\frac{12}{\tau_1} - 4\sqrt{3}\right) \phi' = F$$

has a solution of the form

$$w_0(x, y) = \int_0^x g^*(s, y) ds. \quad (19)$$

We emphasize that $\frac{1}{\tau_1}$ has a singularity at the point $(x, y) = (-\sqrt{3}, 0)$. Therefore we need to be very careful about the behavior of w_0 around this singular point.

Lemma 5 *Suppose η satisfies (11). Then*

$$|w_0| \leq C(1+r)^{\frac{5}{2}}, \text{ for } x \leq 10.$$

Proof. Since η satisfies (11), we have

$$|F_1| \leq (1+r)^{\frac{3}{2}}.$$

Hence using the asymptotic behavior of W and g_1, g_2 , we find that for r large,

$$\begin{aligned} \left| \frac{g_1 F_1}{W \tau_1} \right| &\leq C e^{-\frac{\sqrt{3}}{2}x} (1+r)^{-\frac{1}{2}}, \\ \left| \frac{g_2 F_1}{W \tau_1} \right| &\leq C (1+r)^{-\frac{3}{2}}. \end{aligned}$$

Near the singular point $(-\sqrt{3}, 0)$, using the fact that

$$g_2 - g_1 = O(\tau_1^3),$$

we infer

$$|g^*| \leq C.$$

As a consequence, for $x \leq 10$, we obtain

$$|g^*| \leq C (1+r)^{\frac{3}{2}}.$$

It follows that

$$|w_0| \leq C (1+r)^{\frac{5}{2}}. \quad (20)$$

We remark that as $x \rightarrow +\infty$, since $g_1 = O(\tau_1^2)$, w_0 may not satisfy (20). ■

Lemma 6 *The functions $\xi_0(x, y) := 1$,*

$$\xi_1 := \frac{1}{2}\tau_1^2 - \frac{\sqrt{3}}{6}\tau_1^3,$$

and

$$\xi_2 := \left(\frac{\sqrt{3}}{2}\tau_1 + 1 \right) e^{-\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{4}yi}$$

solve the system

$$\begin{cases} L_1 \phi = 0, \\ M_1 \phi = 0. \end{cases}$$

Proof. This could be checked directly. Alternatively, we can look for solutions of $L_1 \phi = 0$ in the form $f_1(y) \partial_x^{-1} g_1$ and $f_2(y) \partial_x^{-1} g_2$. This reduces to an ODE for the unknown functions f_1 and f_2 . ■

For given function η , we have seen from (19) that the ODE (17) for the unknown function ϕ could be solved for each fixed y . To solve the system (14), we define

$$\Phi_0(x, y) := L_1 \phi - G_1 \phi,$$

and

$$\Phi_1 = \partial_x \Phi_0, \Phi_2 = \partial_x^2 \Phi_0.$$

Note that Φ_i depends on the function ϕ .

Consider the system of equations

$$\begin{cases} \Phi_0(x, y) = 0, \\ \Phi_1(x, y) = 0, \\ \Phi_2(x, y) = 0, \end{cases} \quad \text{for } x = 1. \quad (21)$$

We seek a solution ϕ to (21) in the form $w_0 + w_1$, where

$$w_1(x, y) = \rho_0(y) \xi_0(x, y) + \rho_1(y) \xi_1(x, y) + \rho_2(y) \xi_2(x, y).$$

Lemma 7 *System (21) has a solution (ρ_0, ρ_1, ρ_2) with the initial condition*

$$\rho_i(0) = 0, \quad i = 0, 1, 2.$$

Proof. The equation $\Phi_0 = 0$ could be written as

$$L_1 w_1 = -L_1 w_0 + G_1 \eta := H_0.$$

Similarly, we write $\Phi_1 = 0$ as

$$\partial_x [L_1 w_1] = -\partial_x [L_1 w_0] + \partial_x [G_1 \eta] := H_1.$$

$\Phi_2 = 0$ could be written as

$$\partial_x^2 [L_1 w_1] = -\partial_x^2 [L_1 w_0] + \partial_x^2 [G_1 \eta] := H_2.$$

Consider the system

$$\begin{cases} L_1 w_1 = 0, \\ \partial_x [L_1 w_1] = 0, \\ \partial_x^2 [L_1 w_1] = 0. \end{cases} \quad (22)$$

In view of the definition of w_1 , we know that (22) is a homogeneous system of first order differential equations for the functions ρ_0, ρ_1, ρ_2 . Explicitly, (22) has the form

$$A \begin{pmatrix} \rho'_0 \\ \rho'_1 \\ \rho'_2 \end{pmatrix} = 0,$$

where

$$A = \begin{pmatrix} \frac{i\tau_1 \xi_0}{\sqrt{3}} & \frac{i\tau_1 \xi_1}{\sqrt{3}} & \frac{i\tau_1 \xi_2}{\sqrt{3}} \\ \frac{i\tau_1 \partial_x \xi_0}{\sqrt{3}} & \frac{i\tau_1 \partial_x \xi_1}{\sqrt{3}} & \frac{i\tau_1 \partial_x \xi_2}{\sqrt{3}} \\ \frac{i\tau_1 \partial_x^2 \xi_0}{\sqrt{3}} & \frac{i\tau_1 \partial_x^2 \xi_1}{\sqrt{3}} & \frac{i\tau_1 \partial_x^2 \xi_2}{\sqrt{3}} \end{pmatrix}.$$

Hence (21) has a solution

$$\int_0^y A^{-1}(1, s) \begin{pmatrix} H_0(1, s) \\ H_1(1, s) \\ H_2(1, s) \end{pmatrix} ds.$$

This completes the proof. ■

Proposition 8 Suppose η satisfies (15). Then

$$\begin{aligned}\partial_x^3 \Phi_0 &= \left(-\frac{\sqrt{3}}{2} + \frac{6}{\tau_1} \right) \partial_x^2 \Phi_0 + \frac{1}{\tau_1} \left(2\sqrt{3} - \frac{15}{\tau_1} \right) \partial_x \Phi_0 \\ &\quad + \frac{1}{\tau_1^2} \left(\frac{15}{\tau_1} - 2\sqrt{3} \right) \Phi_0.\end{aligned}$$

Proof. This follows from quite tedious computation using *Mathematica*.

Intuitively, we expect that this identity follows from the compatibility properties of suitable Lax pair of the KP-I equation. But up to now we have not been able to rigorously show this. ■

Lemma 9 The function $\phi = w_0 + w_1$ satisfies (14) for all $(x, y) \in \mathbb{R}^2$.

Proof. By the definition of Φ_0, Φ_1, Φ_2 and Proposition 8, we have

$$\partial_x \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\frac{15}{\tau_1} - 2\sqrt{3}}{\tau_1^2} & \frac{2\sqrt{3} - \frac{15}{\tau_1}}{\tau_1} & \frac{6}{\tau_1} - \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix}.$$

For each fixed $y \neq 0$, by Lemma 7, $\Phi_0(1, y) = \Phi_1(1, y) = \Phi_2(1, y) = 0$. By the uniqueness of solution to ODE, we obtain $\Phi_0(x, y) = \Phi_1(x, y) = \Phi_2(x, y) = 0$, for all $x \in \mathbb{R}$. On the other hand, by the definition of w_0 and w_1 , the second equation of (14) is also satisfied for all $(x, y) \in \mathbb{R}^2$ with $y \neq 0$. By continuity of ϕ , we know that $\phi = w_0 + w_1$ satisfies (14) for all $(x, y) \in \mathbb{R}^2$. This finishes the proof. ■

Lemma 10 Suppose η satisfies (11). Let $\rho_i, i = 0, 1, 2$, be functions given by Lemma 7. Then

$$\rho_1(y) = \rho_2(y) = 0, \text{ for all } y \in \mathbb{R}.$$

Proof. Dividing the equation $\Phi_0 = 0$ by ξ_2 , we get

$$\frac{i}{\sqrt{3}\xi_2} (\rho'_0 \xi_0 + \rho'_1 \xi_1 + \rho'_2 \xi_2) \tau_1 = \frac{1}{\xi_2} H_0. \quad (23)$$

For each fixed $y \in \mathbb{R}$, sending $x \rightarrow -\infty$ in (23) and using the estimate (11), we infer

$$\rho'_2 = 0.$$

This together with the initial condition $\rho_2(0) = 0$ tell us that $\rho_2 = 0$.

Now $\Phi_0 = 0$ becomes

$$\frac{i}{\sqrt{3}} (\rho'_0 \xi_0 + \rho'_1 \xi_1) \tau_1 = H_0.$$

Dividing both sides of this equation by ξ_1 and letting $x \rightarrow -\infty$, we get $\rho'_1 = 0$. Hence $\rho_1 = 0$. The proof is completed. ■

Proof of Proposition 3. We have proved that ρ_1 and ρ_2 are both zero. Now let us define

$$k(y) = \int_{-\infty}^{+\infty} \frac{g_2 F_1}{4\tau_1 W} ds, \text{ for } y \neq 0.$$

Then

$$|w_0 + k(y) \xi_1(x, y)| \leq C(1+r)^{\frac{5}{2}}, \text{ for } x \text{ large.}$$

Using similar arguments as Lemma 10, we see that $k' = 0$. But $k(y) \rightarrow 0$ as $|y| \rightarrow +\infty$. Hence $k(y) = 0$. Then the function $w_0 + \rho_0 \xi_0$ is the desired solution. \blacksquare

4 Linearized Bäcklund transformation between τ_1 and τ_2

In terms of τ_1 and τ_2 , the Bäcklund transformation (10) can be written as

$$\begin{cases} \left(D_x^2 - \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y \right) \tau_1 \cdot \tau_2 = 0, \\ \left(-D_x + i D_y + D_x^3 - \sqrt{3} i D_x D_y \right) \tau_1 \cdot \tau_2 = 0. \end{cases}$$

The linearization of this system is

$$\begin{cases} L_2 \phi = G_2 \eta, \\ M_2 \phi = N_2 \eta. \end{cases} \quad (24)$$

Here

$$\begin{aligned} L_2 \phi &= \left(D_x^2 - \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y \right) \phi \cdot \tau_2, \\ M_2 \phi &= \left(-D_x + i D_y + D_x^3 - \sqrt{3} i D_x D_y \right) \phi \cdot \tau_2, \end{aligned}$$

and

$$\begin{aligned} G_2 \eta &= - \left(D_x^2 - \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y \right) \tau_1 \cdot \eta, \\ N_2 \eta &= - \left(-D_x + i D_y + D_x^3 - \sqrt{3} i D_x D_y \right) \tau_1 \cdot \eta. \end{aligned}$$

Proposition 11 *Let $\eta = \eta(x, y)$ be a function solving the linearized bilinear KP-I equation at τ_2 :*

$$-D_x^2 \eta \tau_2 + D_x^4 \eta \tau_2 = D_y^2 \eta \tau_2. \quad (25)$$

Suppose η satisfies (11). Then the system (14) has a solution ϕ with

$$|\phi| + (|\partial_x \phi| + |\partial_x^2 \phi| + |\partial_x \partial_y \phi|) (1+r) \leq C(1+r)^{\frac{5}{2}}.$$

From the first equation in (24), we get

$$\partial_y \phi \tau_2 = i \left[\sqrt{3} (\partial_x^2 \phi \tau_2 - 2 \partial_x \phi \partial_x \tau_2) - \partial_x \phi \tau_2 \right] + 2i \phi \bar{\tau}_1 - \sqrt{3} i G \eta. \quad (26)$$

Here we have used $\bar{\tau}_1$ to denote the complex conjugate of τ_1 . Inserting (26) into the second equation of (24), we obtain

$$\partial_x^3 \phi \tau_2 + \left(-\frac{\sqrt{3}}{2} \tau_2 - 6x \right) \partial_x^2 \phi + \left(2\sqrt{3}x + \frac{12x^2}{\tau_2} \right) \partial_x \phi - \frac{2\sqrt{3}x}{\tau_2} \bar{\tau}_1 \phi = \frac{F_2}{4}. \quad (27)$$

Here

$$F_2 = N_2 \eta - \left(\frac{6\partial_x \tau_2}{\tau_2} G_2 \eta - 3\partial_x (G_2 \eta) + \sqrt{3} G_2 \eta \right).$$

To solve equation (27), we set

$$\phi = \tau_1 \kappa \text{ and } h = \kappa'.$$

Equation (27) is transformed into the equation

$$\begin{aligned} T(h) &:= \tau_1 \tau_2 h'' + \left[3\tau_2 + \left(-\frac{\sqrt{3}}{2} \tau_2 - 6x \right) \tau_1 \right] h' \\ &+ \left[2 \left(-\frac{\sqrt{3}}{2} \tau_2 - 6x \right) + \tau_1 \left(2\sqrt{3}x + \frac{12x^2}{\tau_2} \right) \right] h \\ &= \frac{F_2}{4}. \end{aligned}$$

Lemma 12 *The homogenous equation*

$$T(h) = 0$$

has two solutions h_1, h_2 , given by

$$\begin{aligned} h_1(x, y) &= (x - yi)^2 + \frac{2}{\sqrt{3}} (x - yi) + 3 + \frac{12(y - \sqrt{3}i) \left(y + \frac{i}{\sqrt{3}} \right)}{\tau_1^2} \\ &= \frac{\tau_2}{3\tau_1^2} \left(3\tau_2 + 4\sqrt{3}(x + \bar{\tau}_1) \right), \end{aligned}$$

and

$$h_2(x, y) = \frac{\tau_2}{\tau_1^2} \left(x + yi - \sqrt{3} \right) e^{\frac{\sqrt{3}}{2}x}.$$

Proof. This equation can be solved using *Mathematica*. ■

Lemma 13 *The system*

$$\begin{cases} \left(D_x^2 - \frac{1}{\sqrt{3}} D_x + \frac{1}{\sqrt{3}} i D_y \right) \phi \cdot \tau_2 = 0, \\ \left(-D_x + i D_y + D_x^3 - \sqrt{3} i D_x D_y \right) \phi \cdot \tau_2 = 0 \end{cases}$$

has three solutions $\zeta_0, \zeta_1, \zeta_2$, given by $\zeta_0 = \tau_1$,

$$\begin{aligned}\zeta_1 &= \tau_1 \partial_x^{-1} h_1 \\ &= \tau_1 \left(\frac{\bar{z}^3}{3} + \frac{\bar{z}^2}{\sqrt{3}} + 3x - 11yi \right) - 12 \left(y - \sqrt{3}i \right) \left(y + \frac{i}{\sqrt{3}} \right),\end{aligned}$$

and

$$\zeta_2 = e^{\sqrt{3}yi} \tau_1 \partial_x^{-1} g_2 = \left(x^2 - \frac{8}{\sqrt{3}}x + y^2 + \frac{4yi}{\sqrt{3}} + 7 \right) e^{\frac{\sqrt{3}}{2}x + \sqrt{3}yi}.$$

Proof. This is similar to Lemma 6 and can be checked by direct computation.

■

Let \tilde{W} be the Wronskian of h_1, h_2 . That is

$$\tilde{W} = h_1 \partial_x h_2 - h_2 \partial_x h_1 = e^{\frac{\sqrt{3}}{2}x} \frac{\tau_2^3}{\tau_1^3}.$$

Variation of parameter formula gives us a solution $\tau_1 \partial_x^{-1} h^*$ to the equation (27), where

$$h^* = h_2 \int_{+\infty}^x \frac{h_1}{\tilde{W}} \frac{F_2}{4\tau_1\tau_2} ds - h_1 \int_{+\infty}^x \frac{h_2}{\tilde{W}} \frac{F_2}{4\tau_1\tau_2} ds.$$

Lemma 14 Suppose η satisfies (11). Let $\tilde{w}_0 = \tau_1 \partial_x^{-1} h^*$. Then

$$|\tilde{w}_0| \leq C(1+r)^{\frac{5}{2}}, \text{ for } x \geq -10.$$

Proof. Since η satisfies (11), we have

$$|F_2| \leq C(1+r)^{\frac{5}{2}}.$$

Hence

$$\begin{aligned}\left| \frac{h_1 F_2}{\tilde{W} \tau_1 \tau_2} \right| &\leq C e^{-\frac{\sqrt{3}x}{2}} (1+r)^{-\frac{3}{2}}, \\ \left| \frac{h_2 F_2}{\tilde{W} \tau_1 \tau_2} \right| &\leq C (1+r)^{-\frac{5}{2}}.\end{aligned}$$

Hence

$$|h^*(x, y)| \leq (1+r)^{\frac{1}{2}}, \text{ for } x \geq -10.$$

It follows that

$$|\tilde{w}_0| \leq C(1+r)^{\frac{5}{2}}, \text{ for } x \geq -10.$$

■

Slightly abusing the notation, we define

$$\Phi_0 = L_2 \phi - G_2 \eta.$$

Then let $\Phi_1 = \partial_x \Phi_0$ and $\Phi_2 = \partial_x^2 \Phi_0$.

We would like to solve the system

$$\begin{cases} \Phi_0 = 0, \\ \Phi_1 = 0, \text{ , for } x = 0. \\ \Phi_2 = 0, \end{cases} \quad (28)$$

Similarly as before, we seek a solution of this problem with the form $\tilde{w}_0 + \tilde{w}_1$, with

$$\tilde{w}_1 = \beta_0(y) \zeta_0 + \beta_1(y) \zeta_1 + \beta_2(y) \zeta_2,$$

where $\beta_0, \beta_1, \beta_2$ are functions of y to be determined.

The problem (28) can be written as

$$\begin{cases} L_2 \tilde{w}_1 = H_0, \\ \partial_x (L_2 \tilde{w}_1) = H_1, \\ \partial_x^2 (L_2 \tilde{w}_1) = H_2. \end{cases} \quad (29)$$

Here

$$\begin{aligned} H_0 &= G_2 \eta - L_2 \tilde{w}_0, \\ H_1 &= \partial_x (G_2 \eta - L_2 \tilde{w}_0), \\ H_2 &= \partial_x^2 (G_2 \eta - L_2 \tilde{w}_0). \end{aligned}$$

Lemma 15 *The equation (29) has a solution $(\beta_0, \beta_1, \beta_2)$ satisfying the initial condition*

$$\beta_i(0) = 0, i = 0, 1, 2.$$

Proof. The proof is similar to that of Lemma 7. ■

Proposition 16 *Suppose η satisfies (25). Then*

$$\begin{aligned} \partial_x^3 \Phi_0 &= \left(\frac{12x}{\tau_2} + \frac{\sqrt{3}}{2} \right) \partial_x^2 \Phi_0 + \left(\frac{6}{\tau_2} - \frac{4\sqrt{3}x}{\tau_2} - \frac{60x^2}{\tau_2^2} \right) \partial_x \Phi_0 \\ &\quad + \left(2\sqrt{3} \frac{x\bar{\tau}_1}{\tau_2^2} - \frac{\sqrt{3}}{\tau_2} - \frac{36x}{\tau_2^2} + \frac{8\sqrt{3}x^2}{\tau_2^2} + \frac{120x^3}{\tau_2^3} \right) \Phi_0. \end{aligned}$$

Proof. We can check this by *Mathematica*. We can also prove this directly by tedious calculation by hand. ■

Lemma 17 *The function $\phi = \tilde{w}_0 + \tilde{w}_1$ solves the system (24) for all $(x, y) \in \mathbb{R}^2$.*

Proof. By Proposition 16,

$$\partial_x \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix},$$

where

$$a_{31} = 2\sqrt{3}\frac{x\bar{\tau}_1}{\tau_2^2} - \frac{\sqrt{3}}{\tau_2} - \frac{36x}{\tau_2^2} + \frac{8\sqrt{3}x^2}{\tau_2^2} + \frac{120x^3}{\tau_2^3},$$

$$a_{32} = \frac{6}{\tau_2} - \frac{4\sqrt{3}x}{\tau_2} - \frac{60x^2}{\tau_2^2},$$

and

$$a_{33} = \frac{12x}{\tau_2} + \frac{\sqrt{3}}{2}.$$

For each fixed y , since $\Phi_i(0, y) = 0, i = 0, 1, 2$, we deduce from the uniqueness of solutions to ODE that $\Phi_i(x, y) = 0$, for all $x \in \mathbb{R}$. ■

Lemma 18 *Let $\beta_i, i = 0, 1, 2$ be the functions given by Lemma 15. Then $\beta_1 = \beta_2 = 0$.*

Proof. The proof is similar to that of Lemma 10. ■

With Lemma 18 at hand, we can prove Proposition 11 similarly as before.

5 Proof of the main theorem

In this section, we will prove Theorem 1.

Lemma 19 *Suppose η satisfies*

$$\begin{cases} L_1\phi = G_1\eta, \\ M_1\phi = N_1\eta, \end{cases}$$

Then

$$-4\partial_x^3\eta + 2\sqrt{3}\partial_x^2\eta = \Theta_1\phi,$$

where

$$\Theta_1\phi := -M_1\phi - \sqrt{3}L_1\phi + 3\partial_x(L_1\phi)$$

In particular, if

$$G_1\eta = N_1\eta = 0,$$

and

$$|\eta| \leq C(1+r)^{\frac{5}{2}}. \quad (30)$$

Then $\eta = c_1 + c_2\tau_1$, for some constants c_1, c_2 .

Proof. The equation $G_1\eta = L_1\phi$ is

$$\left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{i}{\sqrt{3}}D_y\right)\tau_0 \cdot \eta = -L_1\phi.$$

That is,

$$\partial_x^2\eta + \frac{1}{\sqrt{3}}(-\partial_x\eta) + \frac{i}{\sqrt{3}}(-\partial_y\eta) = -L_1\phi.$$

Inserting this identity into the equation

$$\left(-D_x - iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\tau_0 \cdot \eta = -M_1\phi,$$

we get

$$\sqrt{3}\partial_x^2\eta - \partial_x^3\eta - \sqrt{3}i\partial_x\left(-\sqrt{3}i\left(\partial_x^2\eta - \frac{1}{\sqrt{3}}\partial_x\eta + L_1\phi\right)\right) = -M_1\phi - \sqrt{3}L_1\phi.$$

Hence

$$-4\partial_x^3\eta + 2\sqrt{3}\partial_x^2\eta = -M_1\phi - \sqrt{3}L_1\phi + 3\partial_x(L_1\phi).$$

If $L_1\phi = M_1\phi = 0$, then

$$-4\partial_x^3\eta + 2\sqrt{3}\partial_x^2\eta = 0.$$

The solutions of this equation are given by

$$c_1 + c_2x + c_3e^{\frac{\sqrt{3}}{2}x},$$

where c_1, c_2, c_3 are constants which may depend on y . Due to the growth estimate of η , we find that

$$\eta = c_1 + c_2x.$$

Inserting this into the equation $G_1\eta = 0$, we find that η is a linear combination of 1 and τ_1 . ■

Lemma 20 *We have*

$$\Theta_1(x) = 4\sqrt{3}, \quad \Theta_1(y) = 0.$$

and

$$\Theta_1(x^2 - y^2) = 8\sqrt{3}x, \quad \Theta_1(xy) = 2\sqrt{3}i\bar{\tau}.$$

Proof. This follows from direct computation. For instances, we compute

$$\begin{aligned} L_1(x^2 - y^2) &= \left(D_x^2 + \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)(x^2 - y^2) \cdot \tau_1 \\ &= 2\tau_1 - 4x + \frac{1}{\sqrt{3}}(2x\tau_1 - (x^2 - y^2)) + \frac{i}{\sqrt{3}}(-2y\tau_1 - i(x^2 - y^2)) \\ &= \frac{2}{\sqrt{3}}\tau_2. \end{aligned}$$

$$\begin{aligned} M_1(x^2 - y^2) &= \left(-D_x - iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)(x^2 - y^2) \cdot \tau_1 \\ &= -(2x\tau_1 - (x^2 - y^2)) - i(-2y\tau_1 - (x^2 - y^2)i) \\ &\quad + (-3)2 - \sqrt{3}i(-2xi - (-2y)) \\ &= -2\tau_2 - 4\sqrt{3}x. \end{aligned}$$

Then

$$\begin{aligned}\Theta_1(x^2 - y^2) &= 2\tau_2 + 4\sqrt{3}x - \sqrt{3}\frac{2}{\sqrt{3}}\tau_2 + 3\frac{4x}{\sqrt{3}} \\ &= 8\sqrt{3}x.\end{aligned}$$

■

Lemma 21 Define $F(\phi) := (L_1(\phi), M_1(\phi))$, $\mathcal{J}(\phi) := (G_1(\phi), N_1(\phi))$. Then

$$F(x) = \mathcal{J}(x\tau_1 - \sqrt{3}z),$$

$$F(y) = \mathcal{J}(y\tau_1),$$

and

$$F(x^2 - y^2) = \mathcal{J}(\rho_1), \tag{31}$$

$$F(xy) = \mathcal{J}(\rho_2),$$

where

$$\begin{aligned}\rho_1 &= \frac{2}{3}x^3 + \frac{4}{3}\sqrt{3}x^2 - \frac{4\sqrt{3}}{3}yix - \frac{2i}{3}y^3 - \frac{2}{3}\sqrt{3}y^2 - 14yi, \\ \rho_2 &= \frac{1}{2}x^2y + \frac{5}{6}i\sqrt{3}x^2 + \frac{1}{6}ix^3 + \left(\frac{y^2}{2}i + \frac{\sqrt{3}}{3}y\right)x + \frac{y^3}{6} + \frac{\sqrt{3}}{6}y^2i + 5y.\end{aligned}$$

Proof. We only prove (31). The proof of other cases are similar. Consider the equation

$$-4\partial_x^3\eta + 2\sqrt{3}\partial_x^2\eta = \Theta_1(x^2 - y^2) = 8\sqrt{3}x.$$

This equation has a solution

$$\rho_1 = \frac{2}{3}x^3 + \frac{4}{3}\sqrt{3}x^2 + a(y)x + b(y),$$

where $a(y)$ and $b(y)$ are functions to be determined. Since

$$\partial_x^2\rho_1 + \frac{1}{\sqrt{3}}(-\partial_x\rho_1) + \frac{i}{\sqrt{3}}(-\partial_y\rho_1) = L_1(x^2 - y^2) = -2\frac{x^2 + y^2 + 3}{\sqrt{3}},$$

we get

$$4x + \frac{8\sqrt{3}}{3} - \frac{1}{\sqrt{3}}\left(2x^2 + \frac{8}{3}\sqrt{3}x - a\right) - \frac{i}{\sqrt{3}}(a'x + b') = -2\frac{x^2 + y^2 + 3}{\sqrt{3}}.$$

Hence

$$a(y) = -\frac{4\sqrt{3}}{3}yi, b(y) = -\frac{2i}{3}y^3 - \frac{2}{3}\sqrt{3}y^2 - 14yi.$$

Thus

$$\rho_1 = \frac{2}{3}x^3 + \frac{4}{3}\sqrt{3}x^2 - \frac{4\sqrt{3}}{3}yix - \frac{2i}{3}y^3 - \frac{2}{3}\sqrt{3}y^2 - 14yi.$$

■

Lemma 22 Suppose η satisfies

$$\begin{cases} L_2\phi = G_2\eta, \\ M_2\phi = N_2\eta. \end{cases}$$

Then

$$\partial_x^3\eta\tau_1 + \left(\frac{\sqrt{3}}{2}\tau_1 - 3\right)\partial_x^2\eta + \left(\frac{3}{\tau_1} - \sqrt{3}\right)\partial_x\eta + \frac{\sqrt{3}}{\tau_1}\eta = \Theta_2(\phi).$$

Here

$$\Theta_2(\phi) := \frac{6}{\tau_1}L_2\phi - 3\partial_x(L_2\phi) + M_2\phi - \sqrt{3}L_2\phi.$$

Proof. Explicitly, η satisfies

$$\begin{cases} \left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\tau_1 \cdot \eta = -L_2\phi, \\ \left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\tau_1 \cdot \eta = -M_2\phi. \end{cases} \quad (32)$$

Let us write the first equation in this system as

$$iD_y\tau_1 \cdot \eta = -\left(\sqrt{3}D_x^2 - D_x\right)\tau_1 \cdot \eta - \sqrt{3}L_2\phi.$$

That is,

$$\partial_y\eta\tau_1 = -i\left[\partial_x\eta\tau_1 + \sqrt{3}(\partial_x^2\eta\tau_1 - 2\partial_x\eta) - 2\eta\right] - \sqrt{3}iL_2\phi.$$

Inserting this identity into the right hand side of the second equation the system (32), we get the following equation for η :

$$\begin{aligned} & \left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\tau_1 \cdot \eta \\ &= -\sqrt{3}D_x^2\tau_1 \cdot \eta + D_x^3\tau_1 \cdot \eta - \sqrt{3}i[\partial_x\partial_y\eta\tau_1 + \eta\partial_x\partial_y\tau_1 - \partial_x\eta\partial_y\tau_1 - \partial_y\eta\partial_x\tau_1] - \sqrt{3}L_2\phi \\ &= -\sqrt{3}(\partial_x^2\eta\tau_1 - 2\partial_x\eta) + (-\partial_x^3\eta\tau_1 + 3\partial_x^2\eta) - \sqrt{3}\partial_x\eta - \frac{2\sqrt{3}ii}{\tau_1}[\partial_x\eta\tau_1 + \sqrt{3}(\partial_x^2\eta\tau_1 - 2\partial_x\eta) - 2\eta] \\ & \quad - 2\sqrt{3}i\frac{\sqrt{3}iL_2\phi}{\tau_1} - \sqrt{3}\partial_x(\partial_x\eta\tau_1 + \sqrt{3}(\partial_x^2\eta\tau_1 - 2\partial_x\eta) - 2\eta) - 3\partial_x(L_2\phi) - \sqrt{3}L_2\phi \\ &= -4\partial_x^3\eta\tau_1 + \left(-\sqrt{3}\tau_1 + 3 + 6 - \sqrt{3}\tau_1 - 3 + 6\right)\partial_x^2\eta \\ & \quad + \left(2\sqrt{3} - \sqrt{3} + \frac{2\sqrt{3}}{\tau_1}(\tau_1 - 2\sqrt{3}) - \sqrt{3} + 2\sqrt{3}\right)\partial_x\eta - \frac{4\sqrt{3}}{\tau_1}\eta + \frac{6}{\tau_1}L_2\phi - 3\partial_x(L_2\phi) - \sqrt{3}L_2\phi \\ &= -M_2\phi. \end{aligned}$$

Thus η satisfies the following third order ODE:

$$\partial_x^3\eta\tau_1 + \left(\frac{\sqrt{3}}{2}\tau_1 - 3\right)\partial_x^2\eta + \left(\frac{3}{\tau_1} - \sqrt{3}\right)\partial_x\eta + \frac{\sqrt{3}}{\tau_1}\eta = \Theta_2.$$

■

Note that $\eta = \tau_2$ satisfies the homogeneous equation

$$\partial_x^3 \eta \tau_1 + \left(\frac{\sqrt{3}}{2} \tau_1 - 3 \right) \partial_x^2 \eta + \left(\frac{3}{\tau_1} - \sqrt{3} \right) \partial_x \eta + \frac{\sqrt{3}}{\tau_1} \eta = 0. \quad (33)$$

Letting $\eta = \tau_2 \kappa$ and $p = \kappa'$, equation (33) becomes

$$\tau_1 \tau_2 p'' + \left(6x \tau_1 + \left(\frac{\sqrt{3}}{2} \tau_1 - 3 \right) \tau_2 \right) p' + \left(6\tau_1 + 2x \left(\sqrt{3} \tau_1 - 6 \right) + \left(\frac{3}{\tau_1} - \sqrt{3} \right) \tau_2 \right) p = 0. \quad (34)$$

Lemma 23 *The equation (34) has two solutions given by*

$$p_1 = \frac{(x + yi)^2 - 3}{\tau_2^2}$$

and

$$p_2 := \left(8\sqrt{3}x - 4i\sqrt{3}y + 3x^2 + 3y^2 + 21 \right) e^{-\frac{1}{2}\sqrt{3}x \frac{\tau_1}{\tau_2}}.$$

In particular, if η satisfies (30) and

$$G_2 \eta = N_2 \eta = 0,$$

Then $\eta = c_1 z + c_2 \tau_2$.

Note that $\partial_x^{-1} p_1 = -\frac{z}{\tau_2}$. Hence we get a solution $\eta = -z$ for the equation (33). This solution is corresponding to the translation of τ_2 along the x and y axes.

Lemma 24 *We have*

$$\Theta_2(y\tau_1) = i\tau_1\tau_2 + 2i\sqrt{3}x^2 - 2\sqrt{3}xy + 12ix - 2i\sqrt{3}y^2 - 6i\sqrt{3}.$$

$$\begin{aligned} \Theta_2(z^2) &= 2\tau_1\tau_2 - 2\sqrt{3}x^2 - 4i\sqrt{3}xy - 18x + 2\sqrt{3}y^2 - 18iy - 12\sqrt{3} \\ &\quad - \frac{6}{\tau_1} \left(-2x^2 - 4ixy - 2\sqrt{3}x + 2y^2 - 2i\sqrt{3}y + 12 \right). \end{aligned}$$

$$L_2 \rho_1 = -\frac{2}{3}\sqrt{3}x^4 - \frac{4}{9}i\sqrt{3}x^3y + \frac{8}{3}x^3 - \frac{8}{3}ix^2y + \frac{16}{3}\sqrt{3}x^2 + 8x + \frac{2}{9}\sqrt{3}y^4 + \frac{4}{3}iy^3 + 22\sqrt{3}.$$

$$\begin{aligned} M_2 \rho_1 &= -2x^4 - \frac{4}{3}\sqrt{3}x^3 + 4i\sqrt{3}x^2y - 20x^2 \\ &\quad + \frac{8}{3}\sqrt{3}xy^2 + 8ixy - 8\sqrt{3}x + 2y^4 + 28y^2 + 8i\sqrt{3}y + 42. \end{aligned}$$

$$L_2\rho_2 = -\frac{5}{9}\sqrt{3}x^3y - \frac{1}{3}ix^3 - \frac{1}{3}i\sqrt{3}x^2y^2 + 2x^2y + \frac{10}{3}i\sqrt{3}x^2 - \frac{1}{3}\sqrt{3}xy^3 \\ - ix^2y^2 - 2\sqrt{3}xy - ix - \frac{1}{9}i\sqrt{3}y^4 + \frac{2}{3}y^3 + y + 10i\sqrt{3}.$$

$$M_2\rho_2 = -2x^3y - \frac{4}{3}i\sqrt{3}x^3 - \frac{8}{3}\sqrt{3}x^2y - 2ix^2 - 2xy^3 + \frac{2}{3}i\sqrt{3}xy^2 \\ - 30xy - 17i\sqrt{3}x - \frac{2}{3}\sqrt{3}y^3 + 4iy^2 - \sqrt{3}y + 15i.$$

Proof. This follows from Direct computation. For instances,

$$L_2(\rho_1) = \left(D_x^2 - \frac{1}{\sqrt{3}}D_x + \frac{1}{\sqrt{3}}iD_y\right)\rho_1 \cdot \tau_2 \\ = \partial_x^2\rho_1\tau_2 - 4x\partial_x\rho_1 + 2\rho_1 - \frac{1}{\sqrt{3}}(\partial_x\rho_1\tau_2 - 2x\rho_1) + \frac{i}{\sqrt{3}}(\partial_y\rho_1\tau_2 - 2y\rho_1) \\ = \partial_x^2\rho_1\tau_2 + \partial_x\rho_1\left(-4x - \frac{\tau_2}{\sqrt{3}}\right) + \frac{i}{\sqrt{3}}\tau_2\partial_y\rho_1 + \rho_1\left(2 + \frac{2x}{\sqrt{3}} - \frac{2yi}{\sqrt{3}}\right) \\ = -\frac{2}{3}\sqrt{3}x^4 - \frac{4}{9}i\sqrt{3}x^3y + \frac{8}{3}x^3 - \frac{8}{3}ix^2y + \frac{16}{3}\sqrt{3}x^2 + 8x + \frac{2}{9}\sqrt{3}y^4 + \frac{4}{3}iy^3 + 22\sqrt{3}.$$

$$M_2\rho_1 = \left(-D_x + iD_y + D_x^3 - \sqrt{3}iD_xD_y\right)\rho_1 \cdot \tau_2 \\ = \partial_x^3\rho_1\tau_2 + \partial_x^2\rho_1(-6x) + \partial_x\rho_1(-\tau_2 + 6 + 2\sqrt{3}yi) + \partial_y\rho_1(i\tau_2 + 2\sqrt{3}xi) \\ - \partial_x\partial_y\rho_1\sqrt{3}i\tau_2 + \rho_1(2x - 2yi) \\ = -2x^4 - \frac{4}{3}\sqrt{3}x^3 + 4i\sqrt{3}x^2y - 20x^2 \\ + \frac{8}{3}\sqrt{3}xy^2 + 8ixy - 8\sqrt{3}x + 2y^4 + 28y^2 + 8i\sqrt{3}y + 42.$$

The other cases are similar. ■

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let ϕ be a solution of (4) satisfying the assumption of Theorem 1. Then using Lemma 2, we can find η_2 , a solution of (25), satisfying (11). In view of Lemma 22, if $G_2\eta_2 = N_2\eta_2 = 0$. Then $\eta_2 = c_1z + c_2\tau_2$. Therefore, to prove the theorem, from now on, we can assume $G_2\eta_2 \neq 0$ or $N_2\eta_2 \neq 0$.

Using Proposition 11 and the linearization of the identity (8), there exists a solution η_1 of the equation

$$(D_x^2 - D_x^4 + D_y^2)\eta_1 \cdot \tau_1 = 0,$$

satisfying the estimate (11).

Case 1. $G_1\eta_1 = N_1\eta_1 = 0$.

In this case, by Lemma 19, $\eta_1 = a_1 + a_2\tau_1$. Accordingly,

$$\eta_2 = c_1\partial_x\tau_2 + c_2\partial_y\tau_2 + c_3\tau_2.$$

Case 2. $G_1\eta_1 \neq 0$ or $N_1\eta_1 \neq 0$.

In this case, by Proposition 3, there exists a solution η_0 of

$$(D_x^2 - D_x^4 + D_y^2) \eta_0 \cdot \tau_0 = 0, \quad (35)$$

satisfying

$$|\eta_0| + |\partial_x \eta_0| + |\partial_y \eta_0| \leq C(1+r)^{\frac{5}{2}}. \quad (36)$$

From (35) and (36), we infer that

$$\partial_x^2 \eta_0 + \partial_y^2 \eta_0 = 0.$$

Therefore,

$$\eta_0 = c_1 + c_2 x + c_3 y + c_4 (x^2 - y^2) + c_5 xy.$$

We claim that $c_2 = c_3 = c_4 = c_5 = 0$. Indeed, otherwise, using Lemma 24, $\Theta_2(\eta_1)$ would have a term like $x^2 y$ or x^2 . Then using variation of parameter formula and the asymptotic behavior of the solutions p_1, p_2 , we could show that η_2 could not satisfy the estimate (11). This is a contradiction. Hence $\eta_0 = c_0$, a constant. This also implies

$$\eta_2 = c_1 \partial_x \tau_2 + c_2 \partial_y \tau_2 + c_3 \tau_2.$$

The proof is thus finished. ■

References

- [1] M. J. Ablowitz, H. Segur, *Solitons and the inverse scattering transform*. SIAM Studies in Applied Mathematics, 4. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981.
- [2] M. J. Ablowitz, P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*. London Mathematical Society Lecture Note Series, 149. Cambridge University Press, Cambridge, 1991.
- [3] N. G. Berloff, P. H. Roberts, *Motions in a Bose condensate. X. New results on the stability of axisymmetric solitary waves of the Gross-Pitaevskii equation*. J. Phys. A 37 (2004), no. 47, 11333–11351.
- [4] F. Béthuel, P. Gravejat, J. C. Saut, *On the KP-I transonic limit of two-dimensional Gross-Pitaevskii travelling waves*. Dyn. PDE 5(3) (2008), 241–280.
- [5] F. Béthuel, P. Gravejat, J. C. Saut, *Travelling waves for the Gross-Pitaevskii equation. II*. Commun. Math. Phys. 285(2) (2009), 567–591.
- [6] D. Chiron and M. Maris, *Rarefaction pulses for the nonlinear Schrödinger equation in the transonic limit*. Comm. Math. Phys. 326(2014), 329–392.

- [7] A. de Bouard and J. C. Saut, *Solitary waves of generalized Kadomtsev-Petviashvili equations*. Ann. Inst. H. Poincaré-Anal. Non Linéaire 14 (1997), no. 2, 211–236.
- [8] A. de Bouard and J. C. Saut, *Symmetries and decay of the generalized Kadomtsev-Petviashvili solitary waves*. SIAM J. Math. Anal. 28 (1997), no. 5, 1064–1085.
- [9] A. de Bouard, Saut, J. C. *Remarks on the stability of generalized KP solitary waves*. in Mathematical problems in the theory of water waves (Luminy, 1995), 75–84, Contemp. Math., 200, Amer. Math. Soc., Providence, RI, 1996.
- [10] V. S. Dryuma, *On the analytical solution of the two-dimensional Korteweg-de Vries equation*, Sov. Phys. JETP Lett. 19, 753-757 (1974).
- [11] A. S. Fokas, M. J. Ablowitz, *On the inverse scattering and direct linearizing transforms for the Kadomtsev-Petviashvili equation*. Phys. Lett. A 94 (1983), no. 2, 67–70.
- [12] K.A. Gorshkov, D. E. Pelinovsky, Y. A. Stepanyants, *Normal and anormal scattering, formation and decay of bound states of two-dimensional solitons described by the Kadomtsev- Petviashvili equation*. JETP. 77(2), 1993, 237–245.
- [13] R. Hirota, *The direct method in soliton theory*. Translated from the 1992 Japanese original and edited by Atsushi Nagai, Jon Nimmo and Claire Gilson, Cambridge Tracts in Mathematics, 155. Cambridge University Press, Cambridge, 2004.
- [14] M. Hărăguș-Courcelle, R. L. Pego, *Travelling waves of the KP equations with transverse modulations*. C. R. Acad. Sci. Paris. I Math. 328 (1999), no. 3, 227–232.
- [15] X. B. Hu, *Rational solutions of integrable equations via nonlinear superposition formulae*. J. Phys. A 30 (1997), no. 23, 8225–8240.
- [16] C. Jones, S. Putterman and P. H. Roberts, *Motions in a Bose condensate V. Stability of wave solutions of nonlinear Schrödinger equations in two and three dimensions*. J. Phys. A Math. Gen. 19 (1986), 2991–3011.
- [17] C. Jones, P.H. Roberts, *Motion in a Bose condensate IV. Axisymmetric solitary waves*. J. Phys. A Math. Gen. 15 (1982), 2599–619 (1982)
- [18] Kadomtsev, Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Soviet Physics-Doklady, 15, 539-541, 1970.
- [19] Y. Liu, X. P. Wang, *Nonlinear stability of solitary waves of a generalized Kadomtsev-Petviashvili equation*. Comm. Math. Phys. 183 (1997), no. 2, 253–266.

- [20] Z. Lu, E. M. Tian, R. Grimshaw, *Interaction of two lump solitons described by the Kadomtsev-Petviashvili I equation*. Wave Motion 40 (2004), no. 2, 123–135.
- [21] S. V. Manakov, *The inverse scattering transform for the time dependent Schrödinger equation and Kadomtsev-Petviashvili equation*, Physica, D3, 1981, 420–427.
- [22] S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, and V. B. Matveev, *Two dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction*, Physics letter A, 63, 205–206.
- [23] M. Maris, *Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity*. Ann. Math. 178 (2013), 107–182.
- [24] T. Mizumachi, *Asymptotic stability of N -solitary waves of the FPU lattices*. Arch. Ration. Mech. Anal. 207 (2013), no. 2, 393–457.
- [25] T. Mizumachi, R. L. Pego, *Asymptotic stability of Toda lattice solitons*. Nonlinearity 21 (2008), no. 9, 2099–2111.
- [26] A. Nakamura, *Decay mode solution of the two-dimensional KdV equation and the generalized Bäcklund transformation*. J. Math. Phys. 22 (1981), no. 11, 2456–2462.
- [27] C. Rogers, W. F. Shadwick, *Bäcklund transformations and their applications*. Mathematics in Science and Engineering, 161. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982.
- [28] J. Satsuma, M. J. Ablowitz, *Two-dimensional lumps in nonlinear dispersive systems*. J. Math. Phys. 20 (1979), no. 7, 1496–1503.
- [29] J. Villarroel, M. J. Ablowitz, *On the discrete spectrum of the nonstationary Schrödinger equation and multipole lumps of the Kadomtsev-Petviashvili I equation*. Comm. Math. Phys. 207 (1999), no. 1, 1–42.
- [30] X. P. Wang, M. J. Ablowitz, H. Segur, *Wave collapse and instability of solitary waves of a generalized Kadomtsev-Petviashvili equation*. Phys. D 78 (1994), no. 3-4, 241–265.
- [31] X. Zhou, *Inverse scattering transform for the time dependent Schrödinger equation with applications to the KPI equation*. Comm. Math. Phys. 128 (1990), no. 3, 551–564.